

# Z-SET UNKNOTTING IN UNCOUNTABLE PRODUCTS OF REALS

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ABSTRACT. We prove a version of  $Z$ -set unknotting theorem for uncountable products of real numbers.

## 1. INTRODUCTION

Every homeomorphism between  $Z$ -sets of the countable (infinite) power  $R^\omega$  of the real line can be extended to an autohomeomorphism of  $R^\omega$ . This result is widely known and often used in infinite-dimensional topology [1]. Parametric version of this statement is also valid (and used) even though its proof (attributed to H.Toruńczyk) has never been published. In this paper we introduce, for any infinite cardinal  $\tau$ , a concept of  $Z_\tau$ -set and prove, using Toruńczyk's theorem, a fibered version of  $Z_\tau$ -unknotting theorem for uncountable power  $R^\tau$  of the real line (Theorem 4.6). As a corollary we obtain a  $Z_\tau$ -unknotting theorem (Corollary 4.7) for  $R^\tau$ . Similar results (Theorem 4.9 and Corollary 4.10) are valid for the Tychonov cube  $I^\tau$  as well. Using our approach we positively answer (Corollary 5.5) question from [4] whether any homeomorphism between closed  $C$ -embedded  $\sigma$ -compact subsets of  $R^\tau$  admits an extension to an autohomeomorphism of  $R^\tau$ .

## 2. PRELIMINARIES

We refer reader to [2] for needed facts about inverse spectra – in particular, for various versions of the Spectral Theorem. Properties of  $C$ -embedded subsets also can be found there.

Here we collect certain facts from infinite-dimensional topology which will be needed below. By  $\text{cov}(X)$  we denote the collection of all countable functionally open covers of a space  $X$ .

**Definition 2.1.** A closed subset  $Z$  of a space  $X$  is a  $Z$ -set if for each  $\mathcal{U} \in \text{cov}(X)$  there exists a map  $f: X \rightarrow X$  such that  $\text{cl}(f(X)) \cap Z = \emptyset$  and  $f$  is  $\mathcal{U}$ -close to  $\text{id}_X$ .

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**Definition 2.2.** Let  $p: X \rightarrow Y$  be a map. A closed subset  $Z$  of  $X$  is a fibered  $Z$ -set (with respect to  $p$ ) if for each  $\mathcal{U} \in \text{cov}(X)$  there exists a map  $f: X \rightarrow X$  such that  $\text{cl}(f(X)) \cap Z = \emptyset$ ,  $pf = p$  and  $f$  is  $\mathcal{U}$ -close to  $\text{id}_X$ .

As noted above, the following result is due to H. Toruńczyk.

**Theorem 2.1.** *Let  $Z$  and  $F$  be fibered  $Z$ -sets in  $R^\omega \times R^\omega$  with respect to the projection  $\pi_1: R^\omega \times R^\omega \rightarrow R^\omega$ . Then every homeomorphism  $h: Z \rightarrow F$  such that  $\pi_1 h = \pi_1|_Z$  extends to a homeomorphism  $H: R^\omega \times R^\omega \rightarrow R^\omega \times R^\omega$  such that  $\pi_1 H = \pi_1$ .*

A counterpart of the above statement for Hilbert cube fibrations appears in [5].

**Theorem 2.2.** *Let  $Z$  and  $F$  be fibered  $Z$ -sets in  $I^\omega \times I^\omega$  with respect to the projection  $\pi_1: I^\omega \times I^\omega \rightarrow I^\omega$ . Then every homeomorphism  $h: Z \rightarrow F$  such that  $\pi_1 h = \pi_1|_Z$  extends to a homeomorphism  $H: I^\omega \times I^\omega \rightarrow I^\omega \times I^\omega$  such that  $\pi_1 H = \pi_1$ .*

Recall that a map  $f: X \rightarrow Y$  of Polish spaces is soft if for any Polish space  $B$ , its closed subset  $A \subset B$ , and any two maps  $g: A \rightarrow X$ ,  $h: B \rightarrow Y$  with  $fg = h|_A$ , there exists a map  $k: B \rightarrow X$  such that  $g = k|_A$  and  $fk = h$ . In general case (of non-metrizable spaces) we refer reader to [2, Definition 6.1.12].

The following statement will be used below.

**Proposition 2.3.** *Let  $\mathcal{S} = \{X_n, p_n^{n+1}, \omega\}$  be an inverse sequence consisting of Polish spaces and soft projections and  $Z$  be a closed subset of  $\lim \mathcal{S}$ . Suppose that for each  $n \in \omega$  there exists a section  $s_n^{n+1}: X_n \rightarrow X_{n+1}$  of the projection  $p_n^{n+1}: X_{n+1} \rightarrow X_n$  such that  $s_n^{n+1}(X_n) \cap p_{n+1}(Z) = \emptyset$ . Then  $Z$  is a fibered  $Z$ -set in  $X$  with respect to the projection  $p_0: X \rightarrow X_0$ .*

*Proof.* The limit projection  $p_0: X \rightarrow X_0$  is a soft map and  $X$  is a Polish space. For each  $n \in \omega$  equip  $X_n$  with a metric  $d_n$  bounded by  $2^{-n}$ . On  $X$  consider the metric, defined as follows

$$d(\{x_n\}, \{x'_n\}) = \max\{d_n(x_n, x'_n) : n \in \omega\}.$$

Since  $R^\omega$  is an absolute retract there exists a map  $h: R^\omega \times R^\omega \times [0, \infty) \rightarrow R^\omega$  such that  $h(a, b, t) = a$  for each  $t \leq 1$  and  $h(a, b, t) = b$  for each  $t \geq 2$ .

For each  $n \in \omega$  let  $i_{n+1}: X_{n+1} \hookrightarrow R^\omega$  be a closed embedding and note that the diagonal product  $p_n^{n+1} \triangle i_{n+1}: X_{n+1} \rightarrow X_n \times R^\omega$  is also a closed embedding. Consequently, softness of the projection  $p_n^{n+1}: X_{n+1} \rightarrow X_n$  guarantees existence of a map  $r_{n+1}: X_n \times R^\omega \rightarrow X_{n+1}$  such that  $p_n^{n+1} r_{n+1} = \pi_{X_n}$  and  $r_{n+1}|_{(p_n^{n+1} \triangle i_{n+1})(X_{n+1})} = (p_n^{n+1} \triangle i_{n+1})^{-1}$ , where  $\pi_{X_n}: X_n \times R^\omega \rightarrow X_n$  is the projection.

Our goal is to construct, for each  $\alpha: X \rightarrow (0, 1)$ , a map  $g_\alpha: X \rightarrow X$  such that

- (a)  $p_0 g_\alpha = p_0$ ;
- (b)  $d(x, g_\alpha(x)) \leq \alpha(x)$ ;
- (c)  $\text{cl}(g_\alpha(X)) \cap Z = \emptyset$ .

The map  $g_\alpha$  is constructed as the limit  $g_\alpha = \lim\{g_n : n \in \omega\}$  of maps  $g_n : X \rightarrow X_n$ .

Let  $g_0 = p_0$  and suppose that maps for each  $i$ , with  $0 \leq i \leq n$ , we have already constructed a map  $g_i : X \rightarrow X_i$  so that

- (1)<sub>i</sub>  $p_{i-1}^i g_i = g_{i-1}$ ;
- (2)<sub>i</sub> If  $\alpha(x) \leq 2^{-i}$ , then  $g_i(x) = p_i(x)$
- (3)<sub>i</sub> If  $\alpha(x) \geq 2^{-(i-1)}$ , then  $g_i(x) = s_{i-1}^i(p_{i-1}(x))$ ;

Next we define the map  $g_{n+1}$  by letting

$$g_{n+1} = r_{n+1}(g_n \triangle h(i_{n+1} p_{n+1} \triangle i_{n+1} s_n^{n+1} p_n \triangle 2^{n+1} \alpha)).$$

It is clear that conditions (1)<sub>n+1</sub>–(3)<sub>n+1</sub> are satisfied.

Obviously,  $p_n g_\alpha = p_n$  for each  $n \in \omega$ . In particular,  $p_0 g_\alpha = p_0$ . If  $x \in X$ , then there is an index  $n \in \omega$  such that  $2^{-(n+1)} \leq \alpha(x) \leq 2^{-n}$ . Therefore, for each  $i \leq n$ , we conclude, by condition (2)<sub>i</sub>, that  $g_i(x) = p_i(x)$ . This shows that

$$d(x, g_\alpha(x)) = \max\{d_k(p_k(x), p_k(g_\alpha(x))) : k \geq n+1\} \leq 2^{-(n+1)} \leq \alpha(x)$$

and establishes the required closeness of  $\text{id}_X$  and  $g_\alpha$ .

Finally, let us show that  $\text{cl}(g_\alpha(X)) \cap Z = \emptyset$ . Consider a sequence  $\{x_i : i \in \omega\}$  of points in  $X$  such that the sequence  $\{g_\alpha(x_i) : i \in \omega\}$  converges to  $x \in X$ . First observe that  $\inf\{\alpha(x_i) : i \in \omega\} > 0$ . Indeed, if this is not the case, then pick a subsequence  $\{x_{i_j} : j \in \omega\}$  such that  $\lim\{\alpha(x_{i_j}) : j \in \omega\} = 0$ . Since, as shown above,  $d(y, \alpha(y)) \leq \alpha(y)$  for each  $y \in X$ , it follows that  $\lim\{x_{i_j} : j \in \omega\} = x$ . This implies, by continuity of  $\alpha$ , the contradictory equality  $\alpha(x) = 0$ . Consequently,  $0 < \epsilon = \inf\{\alpha(x_i) : i \in \omega\}$ . Take  $n$  large enough so that  $2^{-n} \leq \epsilon$ . Then  $p_{n+1}(g_\alpha(x_i)) = g_{n+1}(x_i) = s_n^{n+1}(p_n(x_i))$  for each  $i \in \omega$  and

$$\begin{aligned} p_{n+1}(x) &= p_{n+1}(\lim\{g_\alpha(x_i) : i \in \omega\}) = \lim\{p_{n+1}(g_\alpha(x_i)) : i \in \omega\} = \\ &= \lim\{s_n^{n+1}(p_n(x_i)) : i \in \omega\} \in s_n^{n+1}(X_n). \end{aligned}$$

Since  $s_n^{n+1}(X_n) \cap p_{n+1}(Z) = \emptyset$  it follows that  $x \notin Z$  and therefore  $\text{cl}(g_\alpha(X)) \cap Z = \emptyset$ .  $\square$

### 3. $Z_\tau$ -SETS

We begin by introducing the following notation

$$B(f, \{\mathcal{U}_t : t \in T\}) = \{g \in C(X, Y) : g \text{ is } \mathcal{U}_t\text{-close to } f \text{ for each } t \in T\},$$

Let  $\tau$  be an infinite cardinal. If  $X$  and  $Y$  are Tychonov spaces then  $C_\tau(X, Y)$  denotes the space of all continuous maps  $X \rightarrow Y$  with the topology defined as follows ([2, p.273]): a set  $G \subseteq C_\tau(X, Y)$  is open if for each  $h \in G$  there exists a collection  $\{\mathcal{U}_t: t \in T\} \subseteq \text{cov}(Y)$ , with  $|T| < \tau$ , such that

$$h \in B(h, \{\mathcal{U}_t: t \in T\}) \subseteq G.$$

Obviously if  $\tau = \omega$ , then the above topology coincides with the limitation topology (see [1]).

In a wide range of situations description of basic neighborhoods in  $C_\tau(Y, X)$  is quite simple. Proof of the following statement follows [2, Lemma 6.5.1] and is therefore omitted.

**Lemma 3.1.** *Let  $\tau > \omega$  and  $X$  be a  $z$ -embedded subspace of a product  $\prod\{X_t: t \in T\}$  of separable metrizable spaces. If  $|T| = \tau$ , then basic neighborhoods of a map  $f: Y \rightarrow X$  in  $C_\tau(Y, X)$  are of the form  $B(f, S) = \{g \in C_\tau(Y, X): \pi_S g = \pi_S f\}$ ,  $S \subseteq T$ ,  $|S| < \tau$ , where  $\pi_S: \prod\{X_t: t \in T\} \rightarrow \prod\{X_t: t \in S\}$  denotes the projection onto the corresponding subproduct.*

Now we are ready to define (fibered)  $Z_\tau$ -sets.

**Definition 3.1.** Let  $\tau \geq \omega$ . A closed subset  $Z \subseteq X$  is a  $Z_\tau$ -set in  $X$  if the set  $\{f \in C_\tau(X, X): f(X) \text{ and } Z \text{ are functionally separated}\}$  is dense in the space  $C_\tau(X, X)$ .

For a map  $p: X \rightarrow Y$  by  $C_\tau^p(X, X)$  we denote set of those  $f \in C_\tau(X, X)$  for which  $pf = p$ .

**Definition 3.2.** Let  $p: X \rightarrow Y$  be a map. A closed subset  $Z$  of  $X$  is a fibered  $Z_\tau$ -set (with respect to  $p$ ) in  $X$  if the set  $\{f \in C_\tau^p(X, X): f(X) \text{ and } Z \text{ are functionally separated}\}$  is dense in the space  $C_\tau^p(X, X)$ .

Obviously in case of compact  $X$  in both definitions it would suffice to require only that  $Z \cap f(X) = \emptyset$ .

#### 4. $Z_\tau$ -SET UNKNOTTING IN $R^\tau$

In this section we prove our main results (Theorem 4.6 and Corollary 4.7). First we need several lemmas. We say that a map  $f: R^A \rightarrow R^A$  (or a map  $f: R^A \times R^A \rightarrow R^A \times R^A$ ) factors through  $C$  by  $f_C$  if there is a map  $f_C: R^C \rightarrow R^C$  (or  $f_C: R^C \times R^C \rightarrow R^C \times R^C$ ) such that  $\pi_C^A f = f_C \pi_C^A$  (or  $(\pi_C^A \times \pi_C^A) f = f_C (\pi_C^A \times \pi_C^A)$ ), where  $\pi_C^A: R^A \rightarrow R^C$  denotes the projection onto the corresponding subproduct. We also say that  $f$  is  $B$ -invariant, where  $B$  is a subset of  $A$ , if  $\pi_B^A f = \pi_B^A$  (or  $(\pi_B^A \times \pi_B^A) f = \pi_B^A \times \pi_B^A$ ).

**Lemma 4.1.** *Let  $|A| > \omega$  and  $f: R^A \times R^A \rightarrow R^A \times R^A$  be a map. Then the set  $\mathcal{M}_f$ , consisting of those  $C \in \exp_\omega A$  for which  $f$  factors through  $C$  is cofinal and  $\omega$ -closed in  $\exp_\omega A$ . If, in addition,  $f$  is  $B$ -invariant for some  $B \subset A$ , then  $f$  factors through  $B \cup C$  whenever  $C \in \mathcal{M}_f$ .*

*Proof.* By Spectral Theorem, the set  $\mathcal{M}_f \subseteq \exp_\omega A$ , consisting of those  $C \in \exp_\omega A$  for which  $f$  factors through  $C$  is cofinal and  $\omega$ -closed in  $\exp_\omega A$ . Straight-forward calculation establishes the second part of the lemma.  $\square$

**Lemma 4.2.** *Let  $|A| > \omega$ ,  $B \subseteq A$  and  $Z$  and  $F$  be closed subsets in  $R^A \times R^A$ . Suppose that  $f, g: R^A \times R^A \rightarrow R^A \times R^A$  are  $B$ -invariant maps such that  $f(Z) = F$ ,  $g(F) = Z$ ,  $gf|Z = \text{id}_Z$  and  $fg|F = \text{id}_F$ . Then the set  $\mathcal{M}_{f,g} = \mathcal{M}_f \cap \mathcal{M}_g$  (see Lemma 4.1) is cofinal and  $\omega$ -closed in  $\exp_\omega A$ . Moreover, if  $Z_{BUC} = \text{cl}_{R^{BUC} \times R^{BUC}}(\pi_{BUC}^A \times \pi_{BUC}^A)(Z)$  and  $F_{BUC} = \text{cl}_{R^{BUC} \times R^{BUC}}(\pi_{BUC}^A \times \pi_{BUC}^A)(F)$ , then for each  $C \in \mathcal{M}_{f,g}$  we have:*

- (1)  $f$  factors through  $B \cup C$  by  $f_{BUC}$ ,
- (2)  $g$  factors through  $B \cup C$  by  $g_{BUC}$ ,
- (3)  $f_{BUC}(Z_{BUC}) = F_{BUC}$ ,
- (4)  $g_{BUC}(F_{BUC}) = Z_{BUC}$ ,
- (5)  $g_{BUC}f_{BUC}|Z_{BUC} = \text{id}_{Z_{BUC}}$ ,
- (6)  $f_{BUC}g_{BUC}|F_{BUC} = \text{id}_{F_{BUC}}$ .

*Proof.* It suffices to note that, by Lemma 4.1 and [2, Proposition 1.1.27], the set  $\mathcal{M}_{f,g} = \mathcal{M}_f \cap \mathcal{M}_g$  is still cofinal and  $\omega$ -closed in  $\exp_\omega A$ . Verification of the listed properties is straightforward.  $\square$

**Lemma 4.3.** *Let  $|B| \geq \omega$ ,  $C = \cup\{C_n: n \in \omega\}$ ,  $|C_n| \leq \omega$ ,  $C_0 = \emptyset$ ,  $Z$  and  $F$  be closed sets in  $R^{BUC}$  and  $f, g: R^{BUC} \rightarrow R^{BUC}$  be  $B$ -invariant maps such that  $f(Z) = F$ ,  $g(F) = Z$ ,  $gf|Z = \text{id}_Z$  and  $fg|F = \text{id}_F$ . Suppose also that for each  $n \in \omega$  there are maps  $s_n^{n+1}, r_n^{n+1}: R^{BUC_n} \rightarrow R^{BUC_{n+1}}$  such that*

- (i)  $\pi_{BUC_n}^{BUC_{n+1}} s_n^{n+1} = \text{id}_{R^{BUC_n}}$ ,
- (ii)  $\pi_{BUC_n}^{BUC_{n+1}} r_n^{n+1} = \text{id}_{R^{BUC_n}}$ ,
- (iii)  $\text{Im}(s_n^{n+1})$  and  $\pi_{BUC_{n+1}}^{BUC}(Z)$  are functionally separated in  $R^{BUC_{n+1}}$ ,
- (iv)  $\text{Im}(r_n^{n+1})$  and  $\pi_{BUC_{n+1}}^{BUC}(F)$  are functionally separated in  $R^{BUC_{n+1}}$ .

*Then there exists a homeomorphism  $H: R^{BUC} \rightarrow R^{BUC}$  such that  $\pi_B^{BUC} H = \pi_B^{BUC}$ , and  $H|Z = f|Z$ .*

*Proof.* Let  $M \subseteq N$ ,  $|N \setminus M| \leq \omega$  and  $\mathcal{M}$  be a cofinal and  $\omega$ -closed subset of  $\exp_\omega M$ . Then the correspondence  $P \mapsto P \cup (N \setminus M)$ ,  $P \in \mathcal{M}$ , identifies  $\mathcal{M}$  with a cofinal and  $\omega$ -closed subset of  $\exp_\omega N$ . Using just described correspondence, Spectral Theorem for factorizing  $\omega$ -spectra [2, Theorem 1.3.6], applied to the maps  $f, g, s_n^{n+1}, r_n^{n+1}$ , and [2, Proposition 1.1.27], we can find a countable subset  $B_0 \subseteq B$  and maps  $\tilde{f}, \tilde{g}: R^{B_0 \cup (C \setminus B)} \rightarrow R^{B_0 \cup (C \setminus B)}$ ,  $\tilde{s}_n^{n+1}, \tilde{r}_n^{n+1}: R^{B_0 \cup (C_n \setminus B)} \rightarrow R^{B_0 \cup (C_{n+1} \setminus B)}$  such that

- (1)  $\pi_{B_0}^{B_0 \cup (C \setminus B)} \tilde{f} = \pi_{B_0}^{B_0 \cup (C \setminus B)};$
- (2)  $\pi_{B_0}^{B_0 \cup (C \setminus B)} \tilde{g} = \pi_{B_0}^{B_0 \cup (C \setminus B)};$
- (3)  $\pi_{B_0 \cup (C \setminus B)}^{B \cup C} f = \tilde{f} \pi_{B_0 \cup (C \setminus B)}^{B \cup C};$
- (4)  $\pi_{B_0 \cup (C \setminus B)}^{B \cup C} g = \tilde{g} \pi_{B_0 \cup (C \setminus B)}^{B \cup C};$
- (5)  $\tilde{f}(\text{cl}(\pi_{B_0 \cup (C \setminus B)}^{B \cup C}(Z))) = \text{cl}(\pi_{B_0 \cup (C \setminus B)}^{B \cup C}(F));$
- (6)  $\tilde{g}(\text{cl}(\pi_{B_0 \cup (C \setminus B)}^{B \cup C}(F))) = \text{cl}(\pi_{B_0 \cup (C \setminus B)}^{B \cup C}(Z));$
- (7)  $\pi_{B_0 \cup (C_{n+1} \setminus B)}^{B \cup C_{n+1}} s_n^{n+1} = \tilde{s}_n^{n+1} \pi_{B_0 \cup (C_n \setminus B)}^{B \cup C_n};$
- (8)  $\pi_{B_0 \cup (C_{n+1} \setminus B)}^{B \cup C_{n+1}} r_n^{n+1} = \tilde{r}_n^{n+1} \pi_{B_0 \cup (C_n \setminus B)}^{B \cup C_n};$
- (9)  $\pi_{B_0 \cup (C_n \setminus B)}^{B_0 \cup (C_{n+1} \setminus B)} \tilde{s}_n^{n+1} = \text{id}_{R^{B_0 \cup (C_n \setminus B)}};$
- (10)  $\pi_{B_0 \cup (C_n \setminus B)}^{B_0 \cup (C_{n+1} \setminus B)} \tilde{r}_n^{n+1} = \text{id}_{R^{B_0 \cup (C_n \setminus B)}};$

By increasing  $B_0$  if necessary and using [2, Corollary 1.1.28], we may assume in addition that

- (11)  $\text{cl}_{R^{B_0 \cup (C_{n+1} \setminus B)}}(\text{Im}(\tilde{s}_n^{n+1})) \cap \text{cl}_{R^{B_0 \cup (C_{n+1} \setminus B)}}(\pi_{B_0 \cup (C_{n+1} \setminus B)}^{B \cup C_{n+1}}(Z)) = \emptyset;$
- (12)  $\text{cl}_{R^{B_0 \cup (C_{n+1} \setminus B)}}(\text{Im}(\tilde{r}_n^{n+1})) \cap \text{cl}_{R^{B_0 \cup (C_{n+1} \setminus B)}}(\pi_{B_0 \cup (C_{n+1} \setminus B)}^{B \cup C_{n+1}}(F)) = \emptyset;$

By Proposition 2.3,  $\text{cl}(\pi_{B_0 \cup (C \setminus B)}^{B \cup C}(Z))$  and  $\text{cl}(\pi_{B_0 \cup (C \setminus B)}^{B \cup C}(F))$  are fibered  $Z$ -set in  $R^{B_0 \cup (C \setminus B)}$  with respect to the projection  $\pi_{B_0}^{B_0 \cup (C \setminus B)}: R^{B_0 \cup (C \setminus B)} \rightarrow R^{B_0}$ . By Theorem 2.1, the homeomorphism

$$\tilde{f}|_{\text{cl}_{R^{B_0 \cup (C \setminus B)}}(\pi_{B_0 \cup (C \setminus B)}^{B \cup C}(Z))}: \text{cl}_{R^{B_0 \cup (C \setminus B)}}(\pi_{B_0 \cup (C \setminus B)}^{B \cup C}(Z)) \rightarrow \text{cl}_{R^{B_0 \cup (C \setminus B)}}(\pi_{B_0 \cup (C \setminus B)}^{B \cup C}(F))$$

can be extended to an autohomeomorphism  $\tilde{H}: R^{B_0 \cup (C \setminus B)} \rightarrow R^{B_0 \cup (C \setminus B)}$  such that  $\pi_{B_0}^{B_0 \cup (C \setminus B)} \tilde{H} = \pi_{B_0}^{B_0 \cup (C \setminus B)}$ . Since the following diagram is a pullback square (note that  $(B \cup C) \setminus B = (B_0 \cup (C \setminus B)) \setminus B_0$ )

$$\begin{array}{ccc} R^{B \cup C} & \xrightarrow{\pi_B^{B \cup C}} & R^B \\ \pi_{B_0 \cup (C \setminus B)}^{B \cup C} \downarrow & & \downarrow \pi_{B_0}^B \\ R^{B_0 \cup (C \setminus B)} & \xrightarrow{\pi_{B_0}^{B_0 \cup (C \setminus B)}} & R^{B_0} \end{array}$$

it follows that there is a unique homeomorphism  $H: R^{B \cup C} \rightarrow R^{B \cup C}$  such that  $\pi_{B_0 \cup (C \setminus B)}^{B \cup C} H = \tilde{H} \pi_{B_0 \cup (C \setminus B)}^{B \cup C}$  and  $\pi_B^{B \cup C} H = \pi_B^{B \cup C}$ . Straightforward verification shows that  $H|Z = f|Z$ .  $\square$

**Lemma 4.4.** *Let  $\{C_n: n \in \omega\}$  be an increasing sequence of countable subsets,  $C_0 = \emptyset$  and  $C = \cup\{C_n: n \in \omega\}$ . Suppose also that  $B$  is an arbitrary set,  $Z$  and  $F$  are closed subsets of  $R^{B \cup C} \times R^{B \cup C}$  and we are given maps  $\varphi_n, \psi_n: R^{B \cup C_n} \times R^{B \cup C_n} \rightarrow R^{B \cup C_n} \times R^{B \cup C_n}$ ,  $n \geq 1$ , such that*

- (1)  $\pi_1^{B \cup C_n} \varphi_n = \pi_1^{B \cup C_n}$ ,  $n \geq 1$ , where  $\pi_1^{B \cup C_n}: R^{B \cup C_n} \times R^{B \cup C_n} \rightarrow R^{B \cup C_n}$  denotes the projection onto the first factor;

- (2)  $\pi_1^{BUC_n} \psi_n = \pi_1^{BUC_n}$ ,  $n \geq 1$ , where  $\pi_1^{BUC_n}: R^{BUC_n} \times R^{BUC_n} \rightarrow R^{BUC_n}$  denotes the projection onto the first factor;
- (3)  $\varphi_{n+1}$  is  $(B \cup C_n)$ -invariant,  $n \in \omega$ ;
- (4)  $\psi_{n+1}$  is  $(B \cup C_n)$ -invariant,  $n \in \omega$ ;
- (5)  $(\pi_{BUC_n}^{BUC} \times \pi_{BUC_n}^{BUC})(Z)$  and  $\text{Im}(\varphi_n)$  are functionally separated in  $R^{BUC_n} \times R^{BUC_n}$ ,  $n \geq 1$ ;
- (6)  $(\pi_{BUC_n}^{BUC} \times \pi_{BUC_n}^{BUC})(F)$  and  $\text{Im}(\psi_n)$  are functionally separated in  $R^{BUC_n} \times R^{BUC_n}$ ,  $n \geq 1$ ;

If  $f, g: R^{BUC} \times R^{BUC} \rightarrow R^{BUC} \times R^{BUC}$  are maps such that

- (7)  $f(Z) = F$ ,  $g(F) = Z$ ;
- (8)  $gf|Z = \text{id}_Z$  and  $fg|F = \text{id}_F$ ;
- (9)  $\pi_1^{BUC} f = \pi_1^{BUC}$ ,  $\pi_1^{BUC} g = \pi_1^{BUC}$ ;
- (10)  $\pi_{BUC_0}^{BUC} f = \pi_{BUC_0}^{BUC}|Z$ ,  $\pi_{BUC_0}^{BUC} g = \pi_{BUC_0}^{BUC}|F$

then the homeomorphism  $h = f|Z: Z \rightarrow F$  can be extended to an autohomeomorphism  $H: R^{BUC} \times R^{BUC} \rightarrow R^{BUC} \times R^{BUC}$  such that  $\pi_1^{BUC} H = \pi_1^{BUC}$  and  $(\pi_B^{BUC} \times \pi_B^{BUC}) H = \pi_B^{BUC} \times \pi_B^{BUC}$ .

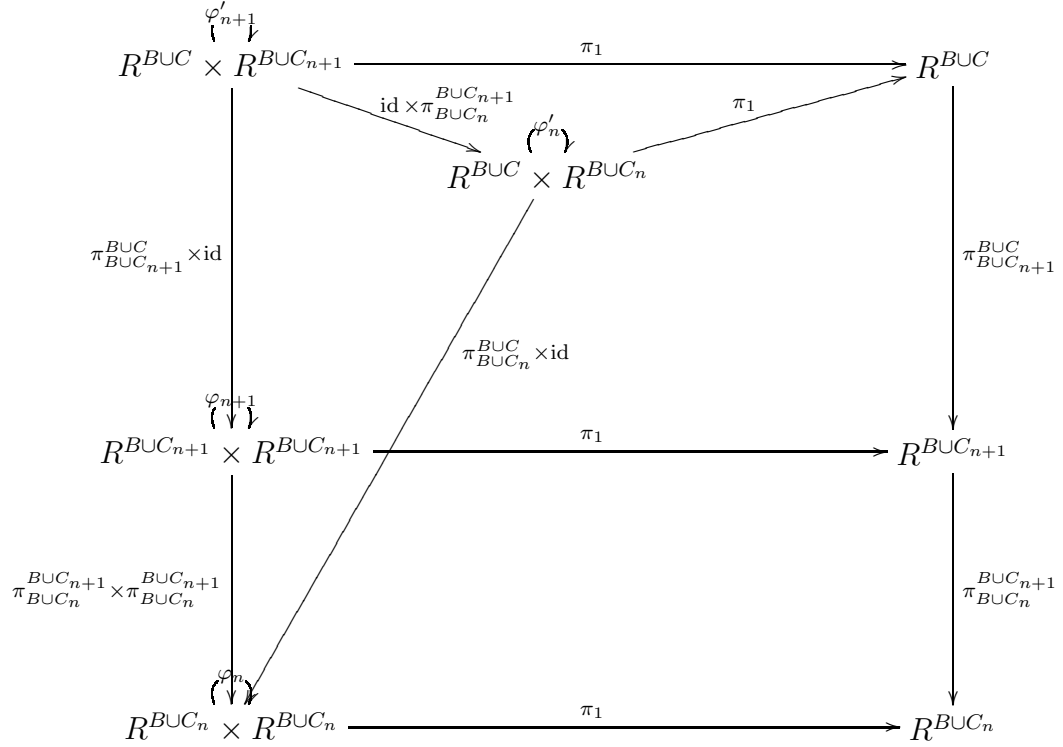
*Proof.* For each  $n \geq 1$  let

$$\varphi'_n = \varphi_n (\pi_{BUC_n}^{BUC} \times \text{id}_{R^{BUC_n}}) \triangle \pi_1: R^{BUC} \times R^{BUC_n} \rightarrow R^{BUC} \times R^{BUC_n}$$

and

$$\psi'_n = \psi_n (\pi_{BUC_n}^{BUC} \times \text{id}_{R^{BUC_n}}) \triangle \pi_1: R^{BUC} \times R^{BUC_n} \rightarrow R^{BUC} \times R^{BUC_n}$$

Here is the corresponding diagram:



It follows from properties (1)–(4) that

$$(11) \quad \left( \text{id}_{R^{BUC}} \times \pi_{BUC_n}^{BUC_{n+1}} \right) \varphi'_{n+1} = \text{id}_{R^{BUC}} \times \pi_{BUC_n}^{BUC_{n+1}}$$

$$(12) \quad \left( \text{id}_{R^{BUC}} \times \pi_{BUC_n}^{BUC_{n+1}} \right) \psi'_{n+1} = \text{id}_{R^{BUC}} \times \pi_{BUC_n}^{BUC_{n+1}}$$

Properties (5) and (6) guarantee that

$$(13) \quad (\text{id}_{R^{BUC}} \times \pi_{BUC_n}^{BUC}) (Z) \text{ and } \text{Im}(\varphi'_n) \text{ are functionally separated in } R^{BUC} \times R^{BUC_n}, n \geq 1;$$

$$(14) \quad (\text{id}_{R^{BUC}} \times \pi_{BUC_n}^{BUC}) (F) \text{ and } \text{Im}(\psi'_n) \text{ are functionally separated in } R^{BUC} \times R^{BUC_n}, n \geq 1;$$

Properties (9) and (10) imply that

$$(15) \quad (\text{id}_{R^{BUC}} \times \pi_{BUC_0}^{BUC}) f = \text{id}_{R^{BUC}} \times \pi_{BUC_0}^{BUC},$$

$$(16) \quad (\text{id}_{R^{BUC}} \times \pi_{BUC_0}^{BUC}) g = \text{id}_{R^{BUC}} \times \pi_{BUC_0}^{BUC}.$$

By Lemma 4.3, applied to the collection  $\{(B \cup C_n) \times (B \cup C_n): n \in \omega\}$ , there exists an autohomeomorphism  $H$  of  $R^{BUC} \times R^{BUC}$  such that  $H|Z = f|Z$  and  $(\text{id}_{R^{BUC}} \times \pi_{BUC_0}^{BUC}) H = \text{id}_{R^{BUC}} \times \pi_{BUC_0}^{BUC}$ . The latter implies that  $\pi_1^{BUC} H = \pi_1^{BUC}$  and  $(\pi_B^{BUC} \times \pi_B^{BUC}) H = \pi_B^{BUC} \times \pi_B^{BUC}$  as required.  $\square$

**Lemma 4.5.** *Let  $\{C_n: n \in \omega\}$  be an increasing sequence of countable subsets,  $C_0 = \emptyset$  and  $C = \cup\{C_n: n \in \omega\}$ . Suppose also that  $B$  is an arbitrary set,  $Z$  and*



$F$  are closed subsets of  $R^{BUC} \times R^{BUC}$  and we are given maps  $\varphi_n, \psi_n: R^{BUC_n} \times R^{BUC_n} \rightarrow R^{BUC_n} \times R^{BUC_n}$ ,  $n \geq 1$ , such that

- (1)  $\pi_1^{BUC_n} \varphi_n = \pi_1^{BUC_n}$ ,  $n \geq 1$ , where  $\pi_1^{BUC_n}: R^{BUC_n} \times R^{BUC_n} \rightarrow R^{BUC_n}$  denotes the projection onto the first factor;
- (2)  $\pi_1^{BUC_n} \psi_n = \pi_1^{BUC_n}$ ,  $n \geq 1$ , where  $\pi_1^{BUC_n}: R^{BUC_n} \times R^{BUC_n} \rightarrow R^{BUC_n}$  denotes the projection onto the first factor;
- (3)  $\left(\pi_{BUC_n}^{BUC_{n+1}} \times \pi_{BUC_n}^{BUC_{n+1}}\right) \varphi_{n+1} = \pi_{BUC_n}^{BUC_{n+1}} \times \pi_{BUC_n}^{BUC_{n+1}}$ ,  $n \in \omega$ ;
- (4)  $\left(\pi_{BUC_n}^{BUC_{n+1}} \times \pi_{BUC_n}^{BUC_{n+1}}\right) \psi_{n+1} = \pi_{BUC_n}^{BUC_{n+1}} \times \pi_{BUC_n}^{BUC_{n+1}}$ ,  $n \in \omega$ ;
- (5)  $(\pi_{BUC_n}^{BUC} \times \pi_{BUC_n}^{BUC})(Z)$  and  $\text{Im}(\varphi_n)$  are functionally separated in  $R^{BUC_n} \times R^{BUC_n}$ ,  $n \geq 1$ ;
- (6)  $(\pi_{BUC_n}^{BUC} \times \pi_{BUC_n}^{BUC})(F)$  and  $\text{Im}(\psi_n)$  are functionally separated in  $R^{BUC_n} \times R^{BUC_n}$ ,  $n \geq 1$ ;

Let also  $H_0: R^{BUC_0} \times R^{BUC_0} \rightarrow R^{BUC_0} \times R^{BUC_0}$  be a homeomorphism such that  $\pi_1^{BUC_0} H = \pi_1^{BUC_0}$  and  $f, g: R^{BUC} \times R^{BUC} \rightarrow R^{BUC} \times R^{BUC}$  be maps such that

- (7)  $f(Z) = F$ ,  $g(F) = Z$ ;
- (8)  $gf|Z = \text{id}_Z$  and  $fg|F = \text{id}_F$ ;
- (9)  $\pi_1^{BUC} f = \pi_1^{BUC}$ ,  $\pi_1^{BUC} g = \pi_1^{BUC}$ ;
- (10)  $(\pi_{BUC_0}^{BUC} \times \pi_{BUC_0}^{BUC}) f = H_0(\pi_{BUC_0}^{BUC} \times \pi_{BUC_0}^{BUC})$ ,  $(\pi_{BUC_0}^{BUC} \times \pi_{BUC_0}^{BUC}) g = H_0^{-1}(\pi_{BUC_0}^{BUC} \times \pi_{BUC_0}^{BUC})$

then the homeomorphism  $h = f|Z: Z \rightarrow F$  can be extended to an autohomeomorphism  $H: R^{BUC} \times R^{BUC} \rightarrow R^{BUC} \times R^{BUC}$  such that  $\pi_1^{BUC} H = \pi_1^{BUC}$  and  $(\pi_B^{BUC} \times \pi_B^{BUC}) H = H_0(\pi_B^{BUC} \times \pi_B^{BUC})$ .

*Proof.* Let  $G_0 = H_0$ ,  $G_{n+1}: R^{BUC_{n+1}} \times R^{BUC_{n+1}} \rightarrow R^{BUC_{n+1}} \times R^{BUC_{n+1}}$  be a homeomorphism such that  $(\pi_{BUC_n}^{BUC_{n+1}} \times \pi_{BUC_n}^{BUC_{n+1}}) G_{n+1} = G_n(\pi_{BUC_n}^{BUC_{n+1}} \times \pi_{BUC_n}^{BUC_{n+1}})$  and  $\pi_1^{BUC_{n+1}} G_{n+1} = \pi_1^{BUC_{n+1}}$ ,  $n \in \omega$ , and  $G = \lim\{G_n: n \in \omega\}$ . Note that  $\pi_1^{BUC} G = \pi_1^{BUC}$  and  $(\pi_{BUC_n}^{BUC} \times \pi_{BUC_n}^{BUC}) G = G_n(\pi_{BUC_n}^{BUC} \times \pi_{BUC_n}^{BUC})$  for each  $n \in \omega$ . In particular,  $(\pi_{BUC_0}^{BUC} \times \pi_{BUC_0}^{BUC}) G = H_0(\pi_{BUC_0}^{BUC} \times \pi_{BUC_0}^{BUC})$ .

Consider the set  $Z' = G(Z)$  and maps  $f' = fG^{-1}$ ,  $g' = Gg: R^{BUC} \times R^{BUC} \rightarrow R^{BUC} \times R^{BUC}$ .

$$f'(Z') = f(G^{-1}(G(Z))) = f(Z) = F \text{ and } g'(F) = G(g(F)) = G(Z) = Z'$$

For any  $x \in Z'$  choose  $y \in Z$  such that  $x = G(y)$ , then

$$g'(f'(x)) = g'(f(G^{-1}(G(y)))) = g'(f(y)) = G(g(f(y))) = G(y) = x$$

similarly if  $x \in F$  then

$$f'(g'(x)) = f'(G(g(x))) = f(G^{-1}(G(g(x)))) = f(g(x)) = x$$

Thus  $g'f'|Z' = \text{id}_{Z'}$  and  $f'g'|F = \text{id}_F$ .

Next note that

$$\pi_1^{BUC} f' = \pi_1^{BUC} f G^{-1} = \pi_1^{BUC} G^{-1} = \pi_1^{BUC}$$

and

$$\pi_1^{BUC} g' = \pi_1^{BUC} G g = \pi_1^{BUC} g = \pi_1^{BUC}$$

Also

$$\begin{aligned} (\pi_{BUC_0}^{BUC} \times \pi_{BUC_0}^{BUC}) f' &= (\pi_{BUC_0}^{BUC} \times \pi_{BUC_0}^{BUC}) f G^{-1} = H_0 (\pi_{BUC_0}^{BUC} \times \pi_{BUC_0}^{BUC}) G^{-1} = \\ &= (\pi_{BUC_0}^{BUC} \times \pi_{BUC_0}^{BUC}) G G^{-1} = \pi_{BUC_0}^{BUC} \times \pi_{BUC_0}^{BUC} \end{aligned}$$

Finally

$$\begin{aligned} (\pi_{BUC_0}^{BUC} \times \pi_{BUC_0}^{BUC}) g' &= (\pi_{BUC_0}^{BUC} \times \pi_{BUC_0}^{BUC}) G g = H_0 (\pi_{BUC_0}^{BUC} \times \pi_{BUC_0}^{BUC}) g = \\ &= H_0 H_0^{-1} (\pi_{BUC_0}^{BUC} \times \pi_{BUC_0}^{BUC}) = \pi_{BUC_0}^{BUC} \times \pi_{BUC_0}^{BUC} \end{aligned}$$

In order to define map  $\varphi'_{n+1}: R^{BUC_{n+1}} \times R^{BUC_{n+1}} \rightarrow R^{BUC_{n+1}} \times R^{BUC_{n+1}}$  we proceed as follows. First let  $L_{n+1}: R^{BUC_{n+1}} \times R^{BUC_n} \rightarrow R^{BUC_{n+1}} \times R^{BUC_n}$  be a homeomorphism such that  $\left(\pi_{BUC_n}^{BUC_{n+1}} \times \text{id}_{R^{BUC_n}}\right) L_{n+1} = G_n \left(\pi_{BUC_n}^{BUC_{n+1}} \times \text{id}_{R^{BUC_n}}\right)$  and  $\pi_1 L_{n+1} = \pi_1$ , where  $\pi_1: R^{BUC_{n+1}} \times R^{BUC_n} \rightarrow R^{BUC_{n+1}}$  is the projection onto the first factor. Let  $\varphi'_{n+1} = G_{n+1} \varphi_{n+1} i L_{n+1}^{-1} \left(\text{id}_{R^{BUC_{n+1}}} \times \pi_{BUC_n}^{BUC_{n+1}}\right)$ , where  $i: R^{BUC_{n+1}} \times R^{BUC_n} \rightarrow R^{BUC_{n+1}} \times R^{BUC_{n+1}}$  is a section of the projection  $\text{id}_{R^{BUC_{n+1}}} \times \pi_{BUC_n}^{BUC_{n+1}}: R^{BUC_{n+1}} \times R^{BUC_{n+1}} \rightarrow R^{BUC_{n+1}} \times R^{BUC_n}$ . Note that  $\left(\text{id}_{R^{BUC_{n+1}}} \times \pi_{BUC_n}^{BUC_{n+1}}\right) G_{n+1} = L_{n+1} \left(\text{id}_{R^{BUC_{n+1}}} \times \pi_{BUC_n}^{BUC_{n+1}}\right)$ .

Since

$$(\pi_{BUC_n}^{BUC} \times \pi_{BUC_n}^{BUC})(Z') = (\pi_{BUC_n}^{BUC} \times \pi_{BUC_n}^{BUC})(G(Z)) = G_n((\pi_{BUC_n}^{BUC} \times \pi_{BUC_n}^{BUC}))(Z)$$

and  $\text{Im}(\varphi'_n) \subseteq G_n(\text{Im}(\varphi_n))$ , it follows that  $(\pi_{BUC_n}^{BUC} \times \pi_{BUC_n}^{BUC})(Z')$  and  $\text{Im}(\varphi'_n)$  are functionally separated in  $R^{BUC_n} \times R^{BUC_n}$ ,  $n \geq 1$ .

$$\begin{aligned}
\pi_1^{BUC_{n+1}} \varphi'_{n+1} &= \pi_1^{BUC_{n+1}} G_{n+1} \varphi_{n+1} i L_{n+1}^{-1} \left( \text{id}_{R^{BUC_{n+1}}} \times \pi_{BUC_n}^{BUC_{n+1}} \right) = \\
&\pi_1 \left( \text{id}_{R^{BUC_{n+1}}} \times \pi_{BUC_n}^{BUC_{n+1}} \right) G_{n+1} \varphi_{n+1} i L_{n+1}^{-1} \left( \text{id}_{R^{BUC_{n+1}}} \times \pi_{BUC_n}^{BUC_{n+1}} \right) = \\
&\pi_1 L_{n+1} \left( \text{id}_{R^{BUC_{n+1}}} \times \pi_{BUC_n}^{BUC_{n+1}} \right) \varphi_{n+1} i L_{n+1}^{-1} \left( \text{id}_{R^{BUC_{n+1}}} \times \pi_{BUC_n}^{BUC_{n+1}} \right) = \\
&\pi_1 L_{n+1} \left( \text{id}_{R^{BUC_{n+1}}} \times \pi_{BUC_n}^{BUC_{n+1}} \right) i L_{n+1}^{-1} \left( \text{id}_{R^{BUC_{n+1}}} \times \pi_{BUC_n}^{BUC_{n+1}} \right) = \\
&\pi_1 L_{n+1} L_{n+1}^{-1} \left( \text{id}_{R^{BUC_{n+1}}} \times \pi_{BUC_n}^{BUC_{n+1}} \right) = \pi_1 \left( \text{id}_{R^{BUC_{n+1}}} \times \pi_{BUC_n}^{BUC_{n+1}} \right) = \pi_1^{BUC_{n+1}}
\end{aligned}$$

and

$$\begin{aligned}
&\left( \pi_{BUC_n}^{BUC_{n+1}} \times \pi_{BUC_n}^{BUC_{n+1}} \right) \varphi'_{n+1} = \\
&\left( \pi_{BUC_n}^{BUC_{n+1}} \times \text{id}_{R^{BUC_n}} \right) \left( \text{id}_{R^{BUC_{n+1}}} \times \pi_{BUC_n}^{BUC_{n+1}} \right) G_{n+1} \varphi_{n+1} i L_{n+1}^{-1} \left( \text{id}_{R^{BUC_{n+1}}} \times \pi_{BUC_n}^{BUC_{n+1}} \right) = \\
&\left( \pi_{BUC_n}^{BUC_{n+1}} \times \text{id}_{R^{BUC_n}} \right) L_{n+1} \left( \text{id}_{R^{BUC_{n+1}}} \times \pi_{BUC_n}^{BUC_{n+1}} \right) \varphi_{n+1} i L_{n+1}^{-1} \left( \text{id}_{R^{BUC_{n+1}}} \times \pi_{BUC_n}^{BUC_{n+1}} \right) = \\
&\left( \pi_{BUC_n}^{BUC_{n+1}} \times \text{id}_{R^{BUC_n}} \right) L_{n+1} \left( \text{id}_{R^{BUC_{n+1}}} \times \pi_{BUC_n}^{BUC_{n+1}} \right) i L_{n+1}^{-1} \left( \text{id}_{R^{BUC_{n+1}}} \times \pi_{BUC_n}^{BUC_{n+1}} \right) = \\
&\left( \pi_{BUC_n}^{BUC_{n+1}} \times \text{id}_{R^{BUC_n}} \right) L_{n+1} L_{n+1}^{-1} \left( \text{id}_{R^{BUC_{n+1}}} \times \pi_{BUC_n}^{BUC_{n+1}} \right) = \\
&\left( \pi_{BUC_n}^{BUC_{n+1}} \times \text{id}_{R^{BUC_n}} \right) \left( \text{id}_{R^{BUC_{n+1}}} \times \pi_{BUC_n}^{BUC_{n+1}} \right) = \pi_{BUC_n}^{BUC_{n+1}} \times \pi_{BUC_n}^{BUC_{n+1}}
\end{aligned}$$

By Lemma 4.4, there exists a homeomorphism  $S: R^{BUC} \times R^{BUC} \rightarrow R^{BUC} \times R^{BUC}$  such that  $\pi_1^{BUC} S = \pi_1^{BUC}$ ,  $(\pi_{BUC_0}^{BUC} \times \pi_{BUC_0}^{BUC}) S = \pi_{BUC_0}^{BUC} \times \pi_{BUC_0}^{BUC}$  and  $S|Z' = f'|Z'$ . It only remains to observe that the homeomorphism  $H = SG$  has all the required properties.  $\square$

**Theorem 4.6.** *Let  $\tau \geq \omega$  and  $Z$  and  $F$  be  $C$ -embedded fibered  $Z_\tau$ -sets of  $R^\tau \times R^\tau$  with respect to the projection  $\pi_1: R^\tau \times R^\tau \rightarrow R^\tau$ . Let also  $h: Z \rightarrow F$  be a homeomorphism such that  $\pi_1 h = \pi_1|Z$ . Then there exists a homeomorphism  $H: R^\tau \times R^\tau \rightarrow R^\tau \times R^\tau$  such that  $H|Z = h$  and  $\pi_1 H = \pi_1$ .*

*Proof.* Let  $|A| = \tau$ . As noted above, for  $\tau = \omega$  statement is known. Below we assume that  $\tau > \omega$ .

Since the projection  $\pi_1^A: R^A \times R^A \rightarrow R^A$  is soft,  $\pi_1^A h = \pi_1^A|Z$  and  $Z$  is  $C$ -embedded in  $R^A$ , it follows from definition of softness [2, Proposition 6.1.16] that there exists a map  $f: R^A \times R^A \rightarrow R^A \times R^A$  such that  $\pi_1^A f = \pi_1^A$  and  $f|Z = h$ . Similarly there is a map  $g: R^A \times R^A \rightarrow R^A \times R^A$  such that  $\pi_1^A g = \pi_1^A$  and  $g|F = h^{-1}$ . Note that since  $\pi_1^A f = \pi_1^A = \pi_1^A g$ , for any  $C \in \mathcal{M}_{f,g}$  in addition to properties listed in Lemma 4.2, we also have  $\pi_1^C f = \pi_1^C = \pi_1^C g$ .

Let  $A = \{a_\alpha : \alpha < \tau\}$  be a well-ordering of  $A$ .

Our first goal is to construct a countable subset  $A_0 \in \mathcal{M}_{f,g}$  such that  $a_0 \in A_0$  and a homeomorphism  $H_0 : R^{A_0} \times R^{A_0} \rightarrow R^{A_0} \times R^{A_0}$  such that  $H_0|_{\text{cl}_{R^{A_0} \times R^{A_0}}(\pi_{A_0}^A \times \pi_{A_0}^A)(Z)} = f_{A_0}|_{\text{cl}_{R^{A_0} \times R^{A_0}}(\pi_{A_0}^A \times \pi_{A_0}^A)(Z)}$ .

Start by choosing  $C_0 \in \mathcal{M}_{f,g}$  such that  $a_0 \in C_0$ . Since  $Z$  and  $F$  are a fibered  $Z_\tau$ -sets, there exist, by Lemma 3.1, maps  $\varphi, \psi : R^A \times R^A \rightarrow R^A \times R^A$  such that

- (1)  $(\pi_{C_0}^A \times \pi_{C_0}^A) \varphi = \pi_{C_0}^A \times \pi_{C_0}^A$ ;
- (2)  $(\pi_{C_0}^A \times \pi_{C_0}^A) \psi = \pi_{C_0}^A \times \pi_{C_0}^A$ ;
- (3)  $\pi_1^A \varphi = \pi_1^A$ ;
- (4)  $\pi_1^A \psi = \pi_1^A$ ;
- (5)  $Z$  and  $\text{Im}(\varphi)$  are functionally separated in  $R^A \times R^A$ ;
- (6)  $F$  and  $\text{Im}(\psi)$  are functionally separated in  $R^A \times R^A$ .

Using Lemmas 4.1 and 4.2, we can find a set  $C_1 \in \mathcal{M}_{f,g} \cap \mathcal{M}_\varphi \cap \mathcal{M}_\psi$  and maps  $\varphi_1, \psi_1 : R^{C_1} \times R^{C_1} \rightarrow R^{C_1} \times R^{C_1}$  such that  $(\pi_{C_1}^A \times \pi_{C_1}^A) \varphi = \varphi_1 (\pi_{C_1}^A \times \pi_{C_1}^A)$  and  $(\pi_{C_1}^A \times \pi_{C_1}^A) \psi = \psi_1 (\pi_{C_1}^A \times \pi_{C_1}^A)$ . It is easy to see that

- (7)  $\pi_1^{C_1} \varphi_1 = \pi_1^{C_1}$ ;
- (8)  $(\pi_{C_0}^{C_1} \times \pi_{C_0}^{C_1}) \varphi_1 = (\pi_{C_0}^{C_1} \times \pi_{C_0}^{C_1})$ ;
- (9)  $\pi_1^{C_1} \psi_1 = \pi_1^{C_1}$ ;
- (10)  $(\pi_{C_0}^{C_1} \times \pi_{C_0}^{C_1}) \psi_1 = (\pi_{C_0}^{C_1} \times \pi_{C_0}^{C_1})$ ;
- (11)  $(\pi_{C_1}^A \times \pi_{C_1}^A)(Z)$  and  $\text{Im}(\varphi_1)$  have disjoint closures in  $R^{C_1} \times R^{C_1}$ ;
- (12)  $(\pi_{C_1}^A \times \pi_{C_1}^A)(F)$  and  $\text{Im}(\psi_1)$  have disjoint closures in  $R^{C_1} \times R^{C_1}$ .

Continuing in this manner we can construct increasing sequence  $\{C_n : n \in \omega\}$  of elements of  $\mathcal{M}_{f,g}$  and maps  $\varphi_n, \psi_n : R^{C_n} \times R^{C_n} \rightarrow R^{C_n} \times R^{C_n}$  satisfying the following properties:

- (i)  $\pi_1^{C_{n+1}} \varphi_{n+1} = \pi_1^{C_{n+1}}$ ;
- (ii)  $(\pi_{C_n}^{C_{n+1}} \times \pi_{C_n}^{C_{n+1}}) \varphi_{n+1} = \pi_{C_n}^{C_{n+1}} \times \pi_{C_n}^{C_{n+1}}$ ;
- (iii)  $\pi_1^{C_{n+1}} \psi_{n+1} = \pi_1^{C_{n+1}}$ ;
- (iv)  $(\pi_{C_n}^{C_{n+1}} \times \pi_{C_n}^{C_{n+1}}) \psi_{n+1} = \pi_{C_n}^{C_{n+1}} \times \pi_{C_n}^{C_{n+1}}$ ;
- (v)  $(\pi_{C_{n+1}}^A \times \pi_{C_{n+1}}^A)(Z)$  and  $\text{Im}(\varphi_{n+1})$  have disjoint closures in  $R^{C_{n+1}} \times R^{C_{n+1}}$ ;
- (vi)  $(\pi_{C_{n+1}}^A \times \pi_{C_{n+1}}^A)(F)$  and  $\text{Im}(\psi_{n+1})$  have disjoint closures in  $R^{C_{n+1}} \times R^{C_{n+1}}$ ;

Let  $A_0 = \cup\{C_n : n \in \omega\}$ . Note that  $A_0 \in \mathcal{M}_{f,g}$ . By Lemma 4.4, the homeomorphism

$$f_{A_0}|_{\dots : \text{cl}_{R^{A_0} \times R^{A_0}}(\pi_{A_0}^A \times \pi_{A_0}^A)(Z)} \rightarrow \text{cl}_{R^{A_0} \times R^{A_0}}(\pi_{A_0}^A \times \pi_{A_0}^A)(F)$$

can be extended to a homeomorphism  $H_0 : R^{A_0} \times R^{A_0} \rightarrow R^{A_0} \times R^{A_0}$  so that  $\pi_1^{A_0} H_0 = \pi_1^{A_0}$ .

Suppose that for each  $\beta < \alpha < \tau$  we have already constructed subsets  $A_\beta \subseteq A$  and autohomeomorphisms  $H_\beta: R^{A_\beta} \times R^{A_\beta} \rightarrow R^{A_\beta} \times R^{A_\beta}$ ,  $\beta < \tau$ , satisfying the following properties:

- (1) $_\beta$   $A_\beta \cup \{a_\beta\} \subseteq A_{\beta+1}$  and  $|A_\beta| < \tau$  for each  $\beta < \alpha$ ;
- (2) $_\beta$   $A_\beta = \cup\{A_\gamma: \gamma < \beta\}$  whenever  $\beta < \alpha$  is a limit ordinal;
- (4) $_\beta$   $\left(\pi_{A_\beta}^{A_{\beta+1}} \times \pi_{A_\beta}^{A_{\beta+1}}\right) H_{\beta+1} = H_\beta \left(\pi_{A_\beta}^{A_{\beta+1}} \times \pi_{A_\beta}^{A_{\beta+1}}\right)$  for each  $\beta < \alpha$ ;
- (5) $_\beta$   $H_\beta = \lim\{H_\gamma: \gamma < \beta\}$  whenever  $\beta < \alpha$  is a limit ordinal;
- (6) $_\beta$   $\left(\pi_{A_\beta}^{A_{\beta+1}} \times \pi_{A_\beta}^{A_{\beta+1}}\right) H_{\beta+1} = H_\beta \left(\pi_{A_\beta}^{A_{\beta+1}} \times \pi_{A_\beta}^{A_{\beta+1}}\right)$  ;
- (7) $_\beta$   $\pi_1^{A_\beta} H_\beta = \pi_1^{A_\beta}$ ;
- (8) $_\beta$   $H_\beta|_{\text{cl}_{\mathbb{R}^{A_\beta} \times R^{A_\beta}} \left(\pi_{A_\beta}^A \times \pi_{A_\beta}^A\right)(Z)} = f_{A_\beta}|_{\text{cl}_{\mathbb{R}^{A_\beta} \times R^{A_\beta}} \left(\pi_{A_\beta}^A \times \pi_{A_\beta}^A\right)(Z)}$  for each  $\beta < \alpha$ .

In order to complete inductive step we need to construct  $A_\alpha$  and  $H_\alpha$ .

Case  $\alpha = \beta + 1$ . Let  $C_{\beta,0} = \emptyset$ . Assuming that  $C_{\beta,n} \in \mathcal{M}_{f,g}$  has already been constructed, we proceed as follows. Since  $|A_\beta \cup C_{\beta,n}| < \tau$  and since  $Z$  and  $F$  are fibered  $Z_\tau$ -sets, there exist maps  $\varphi, \psi: R^A \times R^A \times R^A$  such that  $\pi_1^A \varphi = \pi_1^A = \pi_1^A \psi$ ,  $\left(\pi_{A_\beta \cup C_{\beta,n}}^A \times \pi_{A_\beta \cup C_{\beta,n}}^A\right) \varphi = \pi_{A_\beta \cup C_{\beta,n}}^A \times \pi_{A_\beta \cup C_{\beta,n}}^A = \left(\pi_{A_\beta \cup C_{\beta,n}}^A \times \pi_{A_\beta \cup C_{\beta,n}}^A\right) \psi$ ,  $Z$  and  $\text{Im}(\varphi)$ , as well as  $F$  and  $\text{Im}(\psi)$  are functionally separated in  $R^A \times R^A$ . Let  $C_{\beta,n+1}$  be any element of  $\mathcal{M}_{f,g} \cap \mathcal{M}_\varphi \cap \mathcal{M}_\psi$ , containing  $a_\beta$ . Let  $A_{\beta+1} = A_\beta \cup (\cup\{C_{\beta,n}: n \in \omega\})$ . Lemma 4.5 guarantees that there exists a homeomorphism  $H_{\beta+1}: R^{A_{\beta+1}} \times R^{A_{\beta+1}} \rightarrow R^{A_{\beta+1}} \times R^{A_{\beta+1}}$  satisfying the needed conditions for the ordinal  $\beta + 1$ .

Case of a limit ordinal  $\alpha$ . Let  $A_\alpha = \cup\{A_\beta: \beta < \alpha\}$  and  $H_\alpha = \lim\{H_\beta: \beta < \alpha\}$ .

This completes inductive construction of  $A_\alpha$  and  $H_\alpha$ ,  $\alpha < \tau$ . The required homeomorphism  $H: R^A \times R^A \rightarrow R^A \times R^A$  can now be defined as  $H = \lim\{H_\alpha: \alpha < \tau\}$ .  $\square$

**Corollary 4.7.** *Let  $\tau > \omega$  and  $h: Z \rightarrow F$  be a homeomorphism between  $C$ -embedded  $Z_\tau$ -sets in  $R^\tau$ . Then there is an autohomeomorphism  $H$  of  $R^\tau$  such that  $H|_Z = h$ .*

*Proof.* Let  $a \in R^\tau$ . Note that  $Z_a = \{a\} \times Z$  and  $F_a = \{a\} \times F$  are  $C$ -embedded fibered  $Z_\tau$ -sets in  $R^\tau \times R^\tau$  with respect to the projection  $\pi_1: R^\tau \times R^\tau \rightarrow R^\tau$ . The homeomorphism  $h_a: Z_a \rightarrow F_a$ , defined by letting  $h_a(a, x) = (a, h(x))$ ,  $x \in Z$ , acts fiberwise with respect to the projection  $\pi_1$ . By Theorem 4.6, there is a homeomorphism  $H_a: R^\tau \times R^\tau \rightarrow R^\tau \times R^\tau$  such that  $H_a|_{Z_a} = h_a$  and  $\pi_1 H_a = \pi_1$ . Then the required extension of  $h$  can be defined by letting  $H(x) = H_a(a, x)$ ,  $x \in R^\tau$ .  $\square$

We want to note that Theorem 4.6 remains true for the projection  $\pi_1: X \times R^\tau \rightarrow X$  where  $X$  is an  $R^\tau$ -manifold. Similarly homeomorphisms between  $Z_\tau$ -sets of an  $R^\tau$ -manifold admit extensions to the ambient manifold.

The following statement for compact  $Z$  and  $F$  was proved in [4].

**Corollary 4.8.** *Let  $\tau > \omega$  and  $Z$  and  $F$  be closed  $C$ -embedded  $Z_\tau$ -sets in  $R^\tau$ . Then the following conditions are equivalent:*

- (i)  *$Z$  and  $F$  are homeomorphic;*
- (ii)  *$R^\tau \setminus Z$  and  $R^\tau \setminus F$  are homeomorphic.*

*Proof.* (i)  $\implies$  (ii) follows from Corollary 4.7. In order to prove (ii)  $\implies$  (i) it suffices to note that neither  $Z$  nor  $F$  contains  $G_\delta$ -subsets of  $R^\tau$  and consequently  $R^\tau$  is the Hewitt realcompactification of both  $R^\tau \setminus Z$  and  $R^\tau \setminus F$ . Thus any homeomorphism  $h: R^\tau \setminus F \rightarrow R^\tau \setminus Z$  can be uniquely extended to a homeomorphism  $\tilde{h}: R^\tau \rightarrow R^\tau$ . Then  $\tilde{h}(Z) = F$  and  $\tilde{h}|_Z: Z \rightarrow F$  is the required homeomorphism.  $\square$

**4.1. Compact case.** Precisely same proof (even its simplified version) with just one adjustment – replacing Theorem 2.1 by Theorem 2.2 in the proof of Lemma 4.3 – is needed for obtaining  $Z_\tau$ -set unknotting results in the compact case.

**Theorem 4.9.** *Let  $\tau \geq \omega$  and  $Z$  and  $F$  be fibered  $Z_\tau$ -sets of  $I^\tau \times I^\tau$  with respect to the projection  $\pi_1: I^\tau \times I^\tau \rightarrow I^\tau$ . Let also  $h: Z \rightarrow F$  be a homeomorphism such that  $\pi_1 h = \pi_1|_Z$ . Then there exists a homeomorphism  $H: I^\tau \times I^\tau \rightarrow I^\tau \times I^\tau$  such that  $H|_Z = h$  and  $\pi_1 H = \pi_1$ .*

A version of the following statement appears in [3]

**Corollary 4.10.** *Let  $\tau > \omega$  and  $h: Z \rightarrow F$  be a homeomorphism between  $Z_\tau$ -sets in  $I^\tau$ . Then there is an autohomeomorphism  $H$  of  $I^\tau$  such that  $H|_Z = h$ .*

## 5. RECOGNIZING $Z_\tau$ -SETS

Problem of recognizing  $Z_\tau$ -sets is an important one if one wishes to use unknotting theorems. It is relatively straightforward to detect whether  $\text{id}_{R^\tau}$  can be approximated by self maps images of which miss a given set  $Z$ . It is somewhat harder to find approximating maps whose images are functionally separated from  $Z$ . Of course, in case of the Tychonov cube the latter problem is redundant. We start by addressing the former.

**Lemma 5.1.** *Let  $|A| = \tau > \omega$  and  $Z$  be a closed subset of  $R^A$  containing no closed  $G_\kappa$ -subsets of  $R^A$  for any  $\kappa < \tau$ . Then for each  $B \subseteq A$  with  $|B| < \tau$  there is a section  $i_B^A: R^B \rightarrow R^A$  of the projection  $\pi_B^A: R^A \rightarrow R^B$  such that  $Z \cap i_B^A(R^B) = \emptyset$ .*

*Proof.* Let  $\kappa = \max\{|B|, \omega\}$ ,  $B_0 = B$  and  $x_0 \in R^{B_0}$ . By assumption,  $(\pi_{B_0}^A)^{-1}(x_0) \setminus Z \neq \emptyset$ . Choose a subset  $B_1 \subseteq A$  such that  $B_0 \subseteq B_1$ ,  $|B_1 \setminus B_0| \leq \omega$  and  $(\pi_{B_0}^{B_1})^{-1}(x_0) \setminus \text{cl}_{R^{B_1}}(\pi_{B_1}^A(Z)) \neq \emptyset$ . Let  $x_1 \in (\pi_{B_0}^{B_1})^{-1}(x_0) \setminus \text{cl}_{R^{B_1}}(\pi_{B_1}^A(Z))$ . Choose a section  $i_0^1: R^{B_0} \rightarrow R^{B_1}$  of the projection  $\pi_{B_0}^{B_1}: R^{B_1} \rightarrow R^{B_0}$  such that  $i_0^1(x_0) = x_1$ . Let

$$V_1 = \{x \in R^{B_0} : i_0^1(x) \notin \text{cl}_{R^{B_1}}(\pi_{B_1}^A(Z))\}.$$

Note that  $x_0 \in V_1$  and consequently  $V_1$  is a non-empty open subset of  $R^{B_0}$ .

Let  $\gamma < \kappa^+$ . Suppose that for each  $\lambda$ ,  $1 \leq \lambda < \gamma$ , we have already constructed a subset  $B_\lambda \subseteq A$ , an open subset  $V_\lambda \subseteq R^{B_0}$  and a section  $i_0^\lambda: R^{B_0} \rightarrow R^{B_\lambda}$  of the projection  $\pi_{B_0}^{B_\lambda}: R^{B_\lambda} \rightarrow R^{B_0}$ , satisfying the following conditions:

- (i)  $B_\lambda < B_\mu$ , whenever  $\lambda < \mu < \gamma$ ,
- (ii)  $B_\mu = \cup\{B_\lambda : \lambda < \mu\}$ , whenever  $\mu < \gamma$  is a limit ordinal,
- (iii)  $V_\lambda \subseteq V_\mu$ , whenever  $\lambda < \mu < \gamma$ ,
- (iv)  $V_\mu = \cup\{V_\lambda : \lambda < \mu\}$ , whenever  $\mu < \gamma$  is a limit ordinal,
- (v)  $i_0^\mu = \lim\{i_0^\lambda : \lambda < \mu\}$ , whenever  $\mu < \gamma$  is a limit ordinal,
- (vi)  $i_0^\lambda = \pi_{B_\lambda}^{B_\mu} i_0^\mu$ , whenever  $\lambda < \mu < \gamma$ ,
- (vii)  $V_\lambda = \{x \in R^{B_0} : i_0^\lambda(x) \notin \text{cl}_{R^{B_\lambda}}(\pi_{B_\lambda}^A(Z))\}$

We shall construct a set  $B_\gamma$ , an open subset  $V_\gamma \subseteq R^{B_0}$  and a section  $i_0^\gamma: R^{B_0} \rightarrow R^{B_\gamma}$  of the projection  $\pi_{B_0}^{B_\gamma}: R^{B_\gamma} \rightarrow R^{B_0}$ .

Suppose that  $\gamma$  is a limit ordinal. Then we let  $B_\gamma = \cup\{B_\mu : \mu < \gamma\}$  and

$$i_0^\gamma = \lim\{i_0^\mu : \mu < \gamma\}: R^{B_0} \rightarrow R^{B_\gamma}$$

is well defined and satisfies corresponding conditions (v) and (vi). Let

$$V_\gamma = \{x \in X_{\alpha_0} : i_0^\gamma(x) \notin p_{\alpha_\gamma}(F)\}.$$

Note that  $V_\gamma = \cup\{V_{\alpha_\mu} : \mu < \gamma\}$ .

Next consider the case  $\gamma = \mu + 1$ . Suppose that  $V_\mu \neq R^{B_0}$  and let

$$x_\mu = i_0^\mu(z) \in i_0^\mu(R^{B_0}) \subseteq R^{B_\mu},$$

where  $z \in R^{B_0} \setminus V_\mu$ . By assumption,  $(\pi_{B_\mu}^A)^{-1}(x_\mu) \setminus Z \neq \emptyset$  (note that  $w(X_{\alpha_\mu}) \leq \kappa$ ). Choose a set  $B_\gamma \subseteq A$  so that  $B_\gamma \supseteq B_\mu$ ,  $|B_\gamma \setminus B_\mu| \leq \omega$  and  $(\pi_{B_\mu}^{B_\gamma})^{-1}(x_\mu) \setminus \text{cl}_{R^{B_\gamma}}(\pi_{B_\gamma}^A(Z)) \neq \emptyset$ .

Take any section  $i_\mu^\gamma: R^{B_\mu} \rightarrow R^{B_\gamma}$  of the projection  $\pi_{B_\mu}^{B_\gamma}: R^{B_\gamma} \rightarrow R^{B_\mu}$  such that  $i_\mu^\gamma(x_\mu) = z'$ , where  $z' \in (\pi_{B_\mu}^{B_\gamma})^{-1}(x_\mu) \setminus \text{cl}_{R^{B_\gamma}}(\pi_{B_\gamma}^A(Z))$ . Let  $i_0^\gamma = i_\mu^\gamma i_0^\mu$  and  $V_\gamma = \{x \in R^{B_0} : i_0^\gamma(x) \notin \text{cl}_{R^{B_\gamma}}(\pi_{B_\gamma}^A(Z))\}$ . Note that  $V_\mu \subseteq V_\gamma$  and  $z \in V_\gamma \setminus V_\mu$ . This completes construction of the needed objects in the case  $\gamma = \mu + 1$ .

Thus the construction can be carried out for each  $\lambda < \kappa^+$  and we obtain an increasing collection  $\{V_\lambda : \lambda < \kappa^+\}$  of length  $\kappa^+$  of open subsets of  $R^{B_0}$ . Since  $|B_0| \leq \kappa$ , this collection must stabilize, which means that there is an index  $\lambda_0 < \kappa^+$  such that  $V_\lambda = V_{\lambda_0}$  for any  $\lambda \geq \lambda_0$ . By construction, this is only possible if  $V_{\lambda_0} = R^{B_0}$ . Let  $i_B^A = i_{B_{\lambda_0}}^A i_{B_0}^{B_{\lambda_0}}$ , where  $i_{B_{\lambda_0}}^A : R^{B_{\lambda_0}} \rightarrow R^A$  is any section of the projection  $\pi_{B_{\lambda_0}}^A : R^A \rightarrow R^{B_{\lambda_0}}$ . Clearly  $i_B^A(R^B) \cap Z = \emptyset$ .  $\square$

It turns out that recognizing  $Z_{\omega_1}$ -sets in  $R^{\omega_1}$  is very simple.

**Proposition 5.2.** *The following conditions are equivalent for a closed subset  $Z$  of  $R^{\omega_1}$ :*

- (i)  $Z$  is a  $Z_{\omega_1}$ -set;
- (ii)  $Z$  does not contain  $G_\delta$ -subsets of  $R^{\omega_1}$ .

*Proof.* (i)  $\implies$  (ii). Assuming that  $Z$  does contain a  $G_\delta$ -subset of  $R^{\omega_1}$ , we can find a countable subset  $B \subseteq \omega_1$  and a point  $x_0 \in R^B$  such that  $\pi_B^{-1}(x_0) \subseteq Z$ . On the other hand, by (i), there is a map  $f : R^{\omega_1} \rightarrow R^{\omega_1}$  such that  $\pi_B f = \pi_B$  and  $Z \cap f(R^{\omega_1}) = \emptyset$ . Take a point  $y_0 \in R^{\omega_1}$  such that  $\pi_B(y_0) = x_0$ . Clearly  $f(y_0) \in \pi_B^{-1}(x_0)$  and consequently  $f(y_0) \in Z$ . But this is impossible since  $Z \cap f(R^{\omega_1}) = \emptyset$ .

(ii)  $\implies$  (i). For an arbitrary countable subset  $B \subseteq \omega_1$ , we can find, using Lemma 5.1, a section  $i_B : R^B \rightarrow R^{\omega_1}$  of the projection  $\pi_B : R^{\omega_1} \rightarrow R^B$  such that  $Z \cap i_B(R^B) = \emptyset$ . Since  $i_B(R^B)$  is Lindelöf there is a functionally open subset  $U$  of  $R^{\omega_1}$  such that  $i_B(R^B) \subseteq U$  and  $U \cap Z = \emptyset$ . Since  $i_B(R^B)$  is  $C$ -embedded in  $R^{\omega_1}$  it follows that  $i_B(R^B)$  is functionally separated from  $R^{\omega_1} \setminus U$ . Then the map  $f = i_B \pi_B$  witnesses the fact that  $Z$  is a  $Z_\tau$ -set.  $\square$

In the compact case we have the similar equivalence for any weight.

**Proposition 5.3.** *Let  $\tau > \omega$ . The following conditions are equivalent for a closed subset  $Z$  of  $I^\tau$ :*

- (i)  $Z$  is a  $Z_\tau$ -set;
- (ii)  $Z$  does not contain  $G_\kappa$ -subsets of  $R^\tau$  for any  $\kappa < \tau$ .

**Proposition 5.4.** *Let  $\tau > \omega$ . Any closed  $C$ -embedded Lindelöf subset of  $R^\tau$  is a  $Z_\tau$ -set.*

*Proof.* Let  $|A| = \tau$  and  $Z$  be a closed  $C$ -embedded Lindelöf subset of  $R^\tau$ . Since any  $G_\kappa$ -subset of  $R^A$  with  $\kappa < \tau$  contains a closed copy of  $R^A$  and since  $R^A$  is not Lindelöf, it follows that  $Z$  does not contain any  $G_\kappa$ -subset of  $R^A$ . For any  $B \subseteq A$  with  $|B| < \tau$  we can, according to Lemma 5.1, find a section  $i_B^A : R^B \rightarrow R^A$  of the projection  $\pi_B^A : R^A \rightarrow R^B$  such that  $Z \cap i_B^A(R^B) = \emptyset$ . Since  $i_B^A(R^B)$  is closed in  $R^A$  and since  $Z$  is Lindelöf, there is a functionally open subset  $U$  of  $R^A$  such that  $Z \subseteq U$  and  $U \cap i_B^A(R^B) = \emptyset$ . Since  $Z$  is  $C$ -embedded it follows that  $Z$  is functionally separated from  $R^A \setminus U$ . Consequently  $Z$  and  $i_B^A(R^B)$  are also



functionally separated in  $R^A$ . Thus the map  $f: R^A \rightarrow R^A$ , defined by letting  $f = i_B^A \pi_B^A$ , witnesses the fact that  $Z$  is a  $Z_\tau$ -set.  $\square$

We conclude by positively answering a question from [4]. Proof of the following statement follows from Corollary 4.7 and Proposition 5.4.

**Corollary 5.5.** *Let  $\tau > \omega$ . Any homeomorphism between closed  $C$ -embedded and Lindelöf subsets of  $R^\tau$  can be extended to an autohomeomorphism of  $R^\tau$ .*

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